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Quantization of the Single-qubit Structure with SQUID

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The quantization scheme of a single-qubit structure with Superconducting Quantum Interference Device (SQUID) is given. By introducing a unitary matrix and by means of spectral decomposition, the Hamiltonian operator of the system is exactly formulated in compact forms in spin-1/2 notation. The eigenvalues and eigenstates of the system are discussed.

KEY WORDS: single-qubit; SQUID; quantization; spectral decomposition
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1. INTRODUCTION

A practical quantum computer, if built, would be comprised of a set of coupled two-state quantum systems or quantum bits (qubits) (Pashkin *et al.*, 2003), whose coherent time evolution must be controlled in a computation (Makhlin *et al.*, 1999). Experimentally, nuclear magnetic resonance in molecules (Vandersypen *et al.*, 2001), trapped ions (Gulde *et al.*, 2003), and quantum optical systems (Turchette *et al.*, 1995) have been investigated for embodying quantum computation. These systems have the advantages of high quantum coherence, but cannot be integrated easily to form large-scale circuits.

In recent years, with the development of quantum computation and quantum information, it is well known that, among the variety of qubits implemented, solid-state qubits are of particular interest because of their potential suitability for integrated devices (Pashkin *et al.*, 2003). The superconducting Josephson junction would play an important role in realizing solid-state quantum computation. It has been recognized that suitable Josephson junction devices might serve as qubits in quantum information devices and that quantum logic operations could be

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performed by controlling gate voltages or magnetic fields (Makhlin *et al.*, 1999; Nakamura *et al.*, 1999; Ioffe *et al.*, 1999). The first challenge involved, in the course of realizing quantum computer, is to choose suitable system with Josephson junctions serving as single-qubit. Theoretically and experimentally, the simplest model of Josephson junction qubit was proposed earlier (Bouchiat *et al.*, 1999; Shirman *et al.*, 1997). Subsequently, Makhlin *et al.* put forward the improved ideal design (Makhlin *et al.*, 1999). This design avoided an undesired phase evolution effectively, by tuning the energy splitting between logical states during idle periods between operations. You *et al.* made a further study on this single-qubit structure by considering the role of self-inductance (You *et al.*, 2001), and obtained some instructive conclusions.

In this paper, we introduce a new method to investigate the single-qubit structure with Superconducting Quantum Interference Device (SQUID). Our procedures are as follows: Firstly, we give the number-phase quantization scheme of this single-qubit structure. Secondly, by introducing a unitary matrix and by means of spectral decomposition (Nielsen and Chuang, 2000), we exactly formulate the Hamiltonian operator of the system in compact forms in spin-1/2 representation, from the view of quantum computation, which is important. We also obtain the eigenvalues and eigenstates of the system.

2. NUMBER-PHASE QUANTIZATION OF SINGLE-QUBIT STRUCTURE

The single-qubit structure is drawn in Fig. 1, where a superconducting island is coupled by two Josephson junctions (each with capacitance C_J and Josephson coupling energy E_J) to a segment of a superconducting ring and through a gate

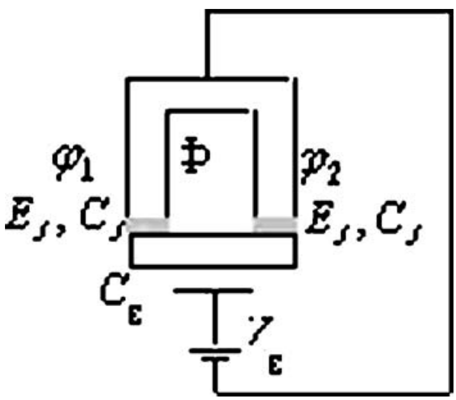

Fig. 1. Single-qubit structure with SQUID.

capacitance C_g to a voltage source V_g. By virtue of Ginzburg–Landau equations, we can deduce

$$\varphi_1 = \varphi(0) + \left(\frac{e}{\hbar}\Phi + \pi m\right), \qquad \varphi_2 = \varphi(0) - \left(\frac{e}{\hbar}\Phi + \pi m\right), \tag{1}$$

where φ_i is the phase difference of the wave function across the Josephson junction, Φ is the magnetic flux through the super-conducting ring, $\varphi(0)$ is initial phase difference and m is an integer number. Supposing $\varphi(0) = 0$ and $m = 0$, it then follows

$$\varphi_1 = \frac{e}{\hbar}\Phi, \qquad \varphi_2 = -\frac{e}{\hbar}\Phi, \tag{2}$$

in Ref. (Feynman *et al.*, 2004), Feynman *et al.* gave

$$\varphi_i = \frac{2eu_i}{\hbar}, \qquad (i = 1, 2), \tag{3}$$

where u_i $(i = 1, 2)$ is the electric potential difference across the Josephson junction. So when tunneling from one side to another, the variation of Cooper-pair's potential energy is

$$-\int_0^t u_i I_c \sin\varphi_i \, dt = E_J(1 - \cos\varphi), \quad (i = 1, 2), \tag{4}$$

where $\varphi = \varphi_1 = -\varphi_2$, $I_c = \frac{2e}{\hbar}E_J$ is the critical electric current of the Josephson junction. The potential energy related to φ of the system is

$$V = \frac{1}{2L}(\Phi - \Phi_x)^2 + 2E_J(1 - \cos\varphi), \tag{5}$$

where L is the self-inductance coefficient of the superconducting ring and Φ_x is the external flux. Substituting Eq. (2) into Eq. (5), we obtain

$$V = \frac{1}{2L}\left(\frac{\hbar}{e}\varphi - \Phi_x\right)^2 + 2E_J(1 - \cos\varphi). \tag{6}$$

The kinetic energy related to $\dot{\varphi}$ is

$$T = \frac{1}{2}(2C_J + C_g)\left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}^2 + N_g\hbar\dot{\varphi} + \frac{2N_g^2e^2}{C_g}, \tag{7}$$

where $N_g = -C_gV_g/(2e)$. Then the Lagrangian of the system can be obtained

$$\mathcal{L} = T - V = \frac{1}{2}(2C_J + C_g)\left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}^2 + N_g\hbar\dot{\varphi} + \frac{2N_g^2e^2}{C_g} - \frac{1}{2L}\left(\frac{\hbar}{e}\varphi - \Phi_x\right)^2 - 2E_J(1 - \cos\varphi). \tag{8}$$

Through the Josephson junction, Cooper-pairs can tunnel from or onto the island. The number n of excess Cooper-pairs on the island depends on the gate voltage. From charge neutrality we have

$$2en = \frac{\hbar}{2e}(2C_{\mathrm{J}} + C_{\mathrm{g}})\dot{\varphi} + 2N_{\mathrm{g}}e, \tag{9}$$

where $2e$ is the charge of a Cooper-pair. The generalized momentum is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\hbar}{2e}\left[\frac{\hbar}{2e}(2C_{\mathrm{J}} + C_{\mathrm{g}})\dot{\varphi} + 2N_{\mathrm{g}}e\right] = n\hbar. \tag{10}$$

Note in particular that the generalized momentum p is proportional to n.

Thus, combining Eq. (8) and Eq. (10), we can obtain the classical Hamiltonian of the system

$$\begin{aligned} H &= p\dot{\varphi} - \mathcal{L} \\ &= 4E_c(n - N_{\mathrm{g}})^2 + 2E_{\mathrm{J}}(1 - \cos\varphi) + \frac{1}{2L}\left(\frac{\hbar}{e}\varphi - \Phi_x\right)^2 - \frac{2N_{\mathrm{g}}^2 e^2}{C_{\mathrm{g}}}, \end{aligned} \tag{11}$$

where $E_c = \frac{e^2}{2(2C_{\mathrm{J}}+C_{\mathrm{g}})}$ is the single-electron charging energy. According to the standard quantization principle, a pair of conjugate quantities n and φ are associated with a pair of Hermitian operators $\hat{n}$ and $\hat{\varphi}$, and they satisfy the commutation relation $[\hat{\varphi}, \hat{n}] = i$. Thus, the single-qubit structure with SQUID is quantized. Then the quantized Hamiltonian of the system can be written as

$$\hat{H} = 4E_c(n - N_{\mathrm{g}})^2 + 2E_{\mathrm{J}}(1 - \cos\varphi) + \frac{1}{2L}\left(\frac{\hbar}{e}\varphi - \Phi_x\right)^2 - \frac{2N_{\mathrm{g}}^2 e^2}{C_{\mathrm{g}}}. \tag{12}$$

3. EIGENVALUES AND EIGENSTATES OF SINGLE-QUBIT STRUCTURE

Now, we consider the system in which the single electron energy is much larger than the Josephson coupling energy, i.e., $E_c \gg E_{\mathrm{J}}$. In this regime, a convenient basis is formed by charge states, parameterized by the excess number of Cooper-pairs n on the island (Makhlin *et al.*, 2001). From the commutation relation $[\hat{n}, \hat{\varphi}] = -i$, we deduce

$$e^{i\varphi} = \sum_n |n+1\rangle\langle n|, \quad e^{-i\varphi} = \sum_n |n\rangle\langle n+1| \tag{13}$$

then we have

$$\cos\varphi = \frac{1}{2}\sum_n (|n\rangle\langle n+1| + |n+1\rangle\langle n|). \tag{14}$$

At the temperature such that $k_{\mathrm{B}}T \ll E_c$ (Bouchiat *et al.*, 1999), we concentrate on such a voltage near a degeneracy point where only two charge states, say $n = 0$

and $n = 1$, play a role, while all other charge states, having a much higher energy, can be ignored (Makhlin *et al.*, 2001). In this case, Eq. (14) can be presented as a matrix

$$\cos\varphi = \frac{1}{2}(|0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \tag{15}$$

Introducing the unitary operator

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{16}$$

$\cos\varphi$ can be diagonalized

$$A = U^{-1}(\cos\varphi)U = \cos(U^{-1}\varphi U) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{17}$$

Consequently, the spectral decomposition (Nielsen and Chuang, 2000) of normal operator A is

$$A = \frac{1}{2}|0\rangle\langle 0| + \left(-\frac{1}{2}\right)|1\rangle\langle 1|. \tag{18}$$

Then we can obtain the operator function of normal operator A

$$\begin{aligned} U^{-1}\varphi U &= \arccos\ A = \arccos(1/2)|0\rangle\langle 0| + \arccos(-1/2)|1\rangle\langle 1| \\ &= \begin{pmatrix} \pi/3 & 0 \\ 0 & 2\pi/3 \end{pmatrix}. \end{aligned} \tag{19}$$

By means of unitary operator U and Eq. (19), the Hamiltonian operator $\hat{H}$ given by Eq. (12) can be presented as a matrix

$$\hat{H}' = U^{-1}\hat{H}U = \begin{pmatrix} H'_{11} & H'_{12} \\ H'_{21} & H'_{22} \end{pmatrix}, \tag{20}$$

where

$$H'_{11} = 2E_c N_g^2 + 2E_c(1 - N_g)^2 - \frac{2}{C_g}N_g^2 e^2 + \frac{\Phi_x^2}{2L} - \frac{\hbar\pi}{3Le}\Phi_x + \frac{\hbar^2\pi^2}{18Le^2} + E_{J,} \tag{21}$$

$$H'_{12} = 2E_c(2N_g - 1), \quad H'_{21} = 2E_c(2N_g - 1), \tag{22}$$

$$H'_{22} = 2E_c N_g^2 + 2E_c(1 - N_g)^2 - \frac{2N_g^2 e^2}{C_g} + \frac{\Phi_x^2}{2L} - \frac{2\hbar\pi}{3Le}\Phi_x + \frac{2\hbar^2\pi^2}{9Le^2} + 3E_J. \tag{23}$$

From Eq. (20), the Hamiltonian operator $\hat{H}$ of the system can be presented in spin-1/2 notation as

$$\hat{H} = \frac{H'_{11} - H'_{22}}{2}\sigma_x + \frac{H'_{12} + H'_{21}}{2}\sigma_z + \frac{H'_{11} + H'_{22}}{2}I. \tag{24}$$

When $N_g \neq 1/2$, Hamiltonian operator given by Eq. (20) has two eigenvalues

$$E_\pm = \frac{1}{2}\left[(H'_{11} + H'_{22}) \pm \sqrt{(H'_{11} + H'_{22})^2 + 4(H'_{12}H'_{21} - H'_{11}H'_{22})}\right], \tag{25}$$

and the corresponding eigenstates are

$$|\chi'_+\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{26}$$

$$|\chi'_-\rangle = -\alpha'|0\rangle + \beta'|1\rangle, \tag{27}$$

where $\alpha^2 = 1 - \beta^2 = H'^2_{12}/[H'^2_{12} + (E_+ - H'_{11})^2], \alpha'^2 = 1 - \beta'^2 = H'^2_{12}/[H'^2_{12} + (E_- - H'_{11})^2]$. Then the eigenstates of the system can be obtained

$$|\chi_+\rangle = U|\chi'_+\rangle = \frac{1}{\sqrt{2}}(\alpha + \beta)|0\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|1\rangle, \tag{28}$$

$$|\chi_-\rangle = U|\chi'_-\rangle = \frac{1}{\sqrt{2}}(\beta' - \alpha')|0\rangle - \frac{1}{\sqrt{2}}(\beta' + \alpha')|1\rangle. \tag{29}$$

Quantum computation requires that single-bit operations (gates) have to be performed (Makhlin *et al.*, 1999). From Eqs. (28) and (29), it is found that the eigenstates of the system are the superposition of two charge states $|0\rangle$and $|1\rangle$, and different states can be obtained by setting gate voltage or external flux Φ_x, which leads to different qubit.

For the special case of $N_g = 1/2$, by a suitable manipulation of the gate voltage V_g, the Hamiltonian operator $\hat{H}'$ takes the simple form

$$\hat{H}' = \begin{pmatrix} H'_{11} & 0 \\ 0 & H'_{22} \end{pmatrix}, \tag{30}$$

whose eigenvalues are $E_1 = H'_{11}$ and $E_2 = H'_{22}$. The corresponding eigenstates are $|\chi'_1\rangle = |0\rangle$ and $|\chi'_2\rangle = |1\rangle$. Thus, the eigenstates of the system are, respectively,

$$|\chi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |\chi_2\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \tag{31}$$

which are known as computational basis states (Nielsen and Chuang, 2000) as same as $|0\rangle$and $|1\rangle$, which are usually used in quantum computation and quantum information.

4. CONCLUSIONS

In this paper, the scheme of number-phase quantization of the single-qubit structure with SQUID is proposed. By introducing a unitary matrix and by means of spectral decomposition, we exactly formulate the Hamiltonian operator of the system in compact forms in spin-1/2 notation. Subsequently, the eigenvalues and the corresponding eigenstates of the system are obtained. It is found that the eigenstates of the system are the superposition of two charge states, $|0\rangle$ and $|1\rangle$, and different states can be obtained by setting gate voltage or external flux Φ_x, which leads to different qubit. In the special case of $N_{\rm g} = 1/2$, two computational basis states, which are usually used in quantum computation and quantum information, are obtained. From the view of quantum computation and quantum information, the above conclusions will be helpful to the study of single-qubit.

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